

# Consensus Problems in Networks of Agents under Nonlinear Protocols with Directed Interaction Topology<sup>1</sup>

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## Abstract

The purpose of this short paper is to provide a theoretical analysis for the consensus problem under nonlinear protocols. A main contribution of this work is to generalize the previous consensus problems under nonlinear protocols for networks with undirected graphs to directed graphs (information flow). Our theoretical result is that if the directed graph is strongly connected and the nonlinear protocol is strictly increasing, then consensus can be realized. Some simple examples are also provided to demonstrate the validity of our theoretical result.

**Key words:** Consensus problems, graph Laplacians, directed graphs, nonlinear protocols.

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# I Introduction

In networks of dynamics agents, “*consensus*” means that all agents need to agree upon certain quantities of interest that depend on their state. A “*consensus protocol*” is an interaction rule that specifies the information exchange between an agent and all of its neighbors on the network, and enables the network to achieve consensus via a process of distributed decision making. Consensus problems (see [1]-[4]) have a long history in the field of computer science, particularly in automata theory and distributed computation. Recently, distributed coordination of networks of dynamic agents has attracted several researchers from various disciplines of engineering and science due to the broad applications of multi-agent systems in many areas, such as collective behavior of flocks and swarms [5, 6], synchronization of coupled oscillators [7]-[9], and so on.

Until now, most papers in the literature mainly concern the consensus problem under linear protocols, with the connection topologies time-varying, state-dependent ( see [1]-[4]). Even in those papers investigating nonlinear protocols, like [2, 3, 9], a strong assumption on networks should be satisfied: the interaction topology should be bidirectional. However, unidirectional communication is important in practical applications and can be easily incorporated, for example, via broadcasting. Also, sensed information flow which plays a central role in schooling and flocking is typically not bidirectional.

So, in this paper, we will look at the consensus problem in networks of dynamic agents, described by ordinary differential equations (ODE), under nonlinear protocols with directed topology. This note can be regarded to extend consensus results under undirected graphs in [2, 3] to the case of directed graphs. Our approach is to model the communication topology as a graph, then by merging spectral graph theory, matrix theory and control theory, we can prove rigorously that if the directed graph is strongly connected and the nonlinear protocol is strictly increasing, then consensus problem can be realized.

An outline of this paper is as follows. In Section II, we define the consensus problem on

graphs. In Section III, we first define the nonlinear protocol, then based on some lemmas of algebraic graph theory and matrix theory, we obtain the main theoretical result. In section IV, two simple examples are also provided to demonstrate the effectiveness of our theoretical result. We conclude this paper in Section V.

## II Consensus problem on graphs

**Definition 1.** (*Weighted Directed Graph*) Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  be a weighted digraph (or directed graph) with the set of nodes  $\mathcal{V} = \{v_1, \dots, v_n\}$ , set of edges  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ , and a weighted adjacency matrix  $\mathcal{A} = (a_{ij})$  with nonnegative adjacency elements  $a_{ij}$ . An edge of  $\mathcal{G}$  is denoted by  $e_{ij} = (v_i, v_j) \in \mathcal{E}$ , which means that node  $v_i$  receives information from node  $v_j$ , and we assume that  $v_i \neq v_j$  for all  $e_{ij}$ , so the graph has no self-loops. The adjacency elements associated with the edges of the graph are positive, i.e.,  $e_{ij} \in \mathcal{E} \iff a_{ij} > 0$ . Moreover, we assume  $a_{ii} = 0$  for all  $i \in 1, \dots, n$ . The set of neighbors of node  $v_i$  is denoted by  $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_i, v_j) \in \mathcal{E}\}$ . The corresponding graph Laplacian  $L = (l_{ij})$  can be defined as

$$l_{ij} = \begin{cases} \sum_{k=1, k \neq i}^n a_{ik}, & i = j \\ -a_{ij}, & i \neq j \end{cases} \quad (1)$$

**Definition 2.** (*Strongly Connected Graph*) A path on a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  of length  $n^* \leq n$  from  $v_{i_0}$  to  $v_{i_{n^*}}$  is an ordered set of distinct vertices  $\{v_{i_0}, \dots, v_{i_{n^*}}\}$  such that  $(v_{i_{j-1}}, v_{i_j}) \in \mathcal{E}$ , for all  $j = 1, \dots, n^*$ . A graph in which a path exists from every vertex to every vertex is said to be strongly connected (SC). Obviously, irreducibility of the graph Laplacian for a graph can imply its strong connectivity.

Without loss of generality, let  $x_i \in R$  denote the value of node  $v_i, i = 1, \dots, n$ . We refer to  $\mathcal{G}_x = (\mathcal{G}, x)$  with  $x = (x_1, \dots, x_n)^T$  as a network (or algebraic graph) with value  $x$  and topology (or information flow)  $\mathcal{G}$ . The value of a node might represent physical quantities including attitude, position, temperature, voltage, and so on.

**Definition 3.** (*Consensus*) Consider a network of dynamic agents with  $\dot{x}_i = u_i$  interested in reaching a consensus via local communication with their neighbors on a graph  $\mathcal{G}_x$ . By reaching a consensus, we mean converging to a one-dimensional agreement space characterized by the following equations:

$$x_1 = x_2 = \cdots = x_n \quad (2)$$

This agreement space can be expressed as  $x = \beta \mathbf{1}$  where  $\mathbf{1} = (1, \cdots, 1)^T$  and  $\beta \in R$  is the collective decision of the group of agents.

**Lemma 1.** (See [1]) Suppose  $L = (l_{ij})$  is a graph Laplacian of a bi-graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$  of  $n$  nodes, i.e.,  $l_{ij} = l_{ji}$ , for any  $i, j \in 1, \cdots, n$ . The following sum-of-squares (SOS) property holds, for any  $x = (x_1, \cdots, x_n)^T$ ,

$$x^T L x = - \sum_{j>i} l_{ij} (x_j - x_i)^2 \quad (3)$$

### III Consensus Analysis

#### A. Nonlinear consensus protocol

In this paper, we propose the following nonlinear consensus protocol  $h(\cdot) : R \rightarrow R$  to solve consensus problems in a network of continuous-time integrator agents with fix connection topology  $\mathcal{G}_x$ :

$$\dot{x}_i(t) = \sum_{j \in \mathcal{N}_i} a_{ij} \left( h(x_j(t)) - h(x_i(t)) \right), \quad i = 1, \cdots, n \quad (4)$$

If  $L = (l_{ij})$  is the corresponding graph Laplacian of  $\mathcal{G}_x$  defined in Definition 1, then the above equations also can be rewritten as

$$\dot{x}_i(t) = - \sum_{j=1}^n l_{ij} h(x_j(t)), \quad i = 1, \cdots, n \quad (5)$$

Throughout this paper, we assume that  $h(\cdot)$  is a strictly increasing function. Without loss of generality, we assume  $h(0) = 0$ .

## B. Algebraic graph theory and matrix theory

In this part, we introduce some basic concepts, notations and lemmas in algebraic graph theory and matrix theory that will be used throughout this paper.

**Lemma 2.** (*Spectral localization. See [3]*) *Let  $\mathcal{G}$  be a strongly connected digraph of  $n$  nodes. Then  $\text{rank}(L) = n - 1$ , and all nontrivial eigenvalues of  $L$  have positive real part.*

**Remark 1.** *Lemma 2 holds under a weaker condition of existence of a directed spanning tree for  $\mathcal{G}$ .  $\mathcal{G}$  has a directed spanning tree if there exists a node  $r$  (root) such that all other nodes can be linked to  $r$  via a directed path (see relating papers [4, 8]). In fact, in digraphs with spanning tree (leader-follower model), the root node is commonly known as a leader, which does not receive any information from other nodes.*

**Lemma 3.** *Assume  $\mathcal{G}$  is a strongly connected digraph with graph Laplacian  $L$ , then ([10])*

1.  $\mathbf{1} = (1, 1, \dots, 1)^T$  is the right eigenvector of  $L$  corresponding to eigenvalue 0 with multiplicity 1;

2. Let  $\xi = (\xi_1, \xi_2, \dots, \xi_n)^T$  be the left eigenvector of  $L$  corresponding to the eigenvalue 0. Then,  $\xi_i > 0$ ,  $i = 1, 2, \dots, n$ ; and its multiplicity is 1. In the following, we always assume  $\sum_{i=1}^n \xi_i = 1$ .

## C. Main results

In this part, we will give a theorem, which shows that if the directed graph is strongly connected and the nonlinear function is strictly increasing, then the consensus problem can be realized.

**Theorem 1.** *Suppose the digraph  $\mathcal{G}_x$  is a strongly connected.  $L$  is the corresponding graph Laplacian in Definition 1. Then consensus can be realized globally for all initial states by the nonlinear protocol (5) and the group decision is  $x_\xi = \sum_{i=1}^n \xi_i x_i(0)$ , where  $\xi = (\xi_1, \dots, \xi_n)^T$  is defined in Lemma 3.*

Before the proof of Theorem 1, we introduce a reference node (or virtue leader)  $x_\xi(t) = \sum_{i=1}^n \xi_i x_i(t)$ . It is clearly

$$\dot{x}_\xi(t) = \sum_{i=1}^n \xi_i \dot{x}_i(t) = - \sum_{i=1}^n \xi_i \sum_{j=1}^n l_{ij} h(x_j(t)) = - \sum_{j=1}^n h(x_j(t)) \sum_{i=1}^n \xi_i a_{ij} = 0 \quad (6)$$

Therefore, we obtain the following simple but useful proposition, which plays an important role in the discussion of final group decision.

**Proposition 1.**  $x_\xi(t)$  is time-invariant for the network (5), i.e.,  $x_\xi = \sum_{i=1}^n \xi_i x_i(0) = x_\xi(t)$  for all  $t \geq 0$ .

**Proof of Theorem 1** Denote  $x_\xi = \sum_{i=1}^n \xi_i x_i(0)$ ,  $x(t) = (x_1(t), \dots, x_n(t))^T$ , and  $H(x(t)) = (h(x_1(t)), \dots, h(x_n(t)))^T$ . Then equations (5) can be rewritten in the compact form as

$$\dot{x}(t) = -LH(x(t)) \quad (7)$$

Define a function as:

$$V(x(t)) = \sum_{i=1}^n \xi_i \int_0^{x_i(t)} h(s) ds \quad (8)$$

Obviously,  $V(x(t)) \geq 0$  is radially unbounded, and  $V(x(t)) = 0$  if and only if  $x(t) = 0$ .

Denote  $B = (b_{ij}) = (\Xi L + L^T \Xi)/2$ , where  $\Xi = \text{diag}(\xi)$ . It is easy to check that  $B$  is a symmetric matrix with zero row-sum, i.e.,  $B$  can be regarded as a graph Laplacian of a bi-graph. Differentiating  $V(x(t))$  and using Lemma 1, we have

$$\begin{aligned} \dot{V}(x(t)) &= - \sum_{i=1}^n \xi_i h(x_i(t)) \sum_{j \in \mathcal{N}_i} l_{ij} h(x_j(t)) \\ &= -H(x(t))^T \Xi L H(x(t)) = -H(x(t))^T B H(x(t)) \\ &= \sum_{i>j} b_{ij} (h(x_i(t)) - h(x_j(t)))^2 \leq 0, \quad (\text{since } b_{ij} \leq 0) \end{aligned} \quad (9)$$

Thus,  $0 \leq V(x(t)) \leq V(x(0))$ , which implies  $x(t)$  is bounded for any  $t \geq 0$ . And the largest invariant subset ( $\Omega$ -limit set) for the equations (5) is

$$\Omega = \{x : x_i(t) = x_j(t); i, j = 1, \dots, n\} \quad (10)$$

Now, we claim that for all  $i = 1, \dots, n$ ,  $\lim_{t \rightarrow \infty} x_i(t) = x_\xi$ .

In fact, if  $t_m \rightarrow \infty$  and for all  $i, j = 1, \dots, n$ ,  $x_i(t_m) \rightarrow \beta$ , then,

$$x_\xi = \lim_{t \rightarrow \infty} \sum_{i=1}^n \xi_i x_i(t_m) = \sum_{i=1}^n \xi_i \beta = \beta \quad (11)$$

which means that  $x_\xi$  is the group decision of the consensus problem. Theorem 1 is proved completely.

**Remark 2.** Let  $h(x_i(t)) = \alpha x_i(t)$  with  $\alpha > 0$ , The nonlinear function  $h(\cdot)$  becomes a linear function. therefore, Theorem 1 can be regarded as a generalization of the consensus problem under linear protocols. It also give a simple proof for the consensus problem under linear protocols, too.

**Remark 3.** Assume  $(h(w_1) - h(w_2)) / (w_1 - w_2) \geq \alpha$  holds for  $\alpha > 0$  and any  $w_1 \neq w_2 \in R$ . In this case, we have

$$\dot{V}(x(t)) = \sum_{i>j} b_{ij} (h(x_i(t)) - h(x_j(t)))^2 \leq \alpha^2 \sum_{i>j} b_{ij} (x_i(t) - x_j(t))^2 \quad (12)$$

and the consensus problem will be realized exponentially. Moreover, it seems that the nonlinear protocol can be realized faster than that under the linear protocol  $h(w) = \alpha w$ . Therefore, nonlinear protocols can be applied to calculate the average value of large-scale networks more effectively.

## IV Numerical examples

In this section, we give two numerical simulations to verify the validity of our theory.

Consider a network of a strongly connected digraph  $\mathcal{G}_x$  with 3 agents

$$\dot{x}_i(t) = - \sum_{j=1}^3 l_{ij} h(x_j(t)), \quad i = 1, 2, 3$$

where  $x_i(t) \in \mathbb{R}$  and the Laplacian of  $\mathcal{G}_x$  is

$$L = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \quad (13)$$

Its left eigenvector with eigenvalue 0 is  $\xi = (1/4, 1/4, 1/2)^T$ .

**Example 1:** In this simulation, the nonlinear protocol is assumed as  $h(x_i(t)) = \alpha x_i(t) + \sin(x_i(t))$ ,  $i = 1, 2, 3$ . The initial value is taken as:  $(x_1(0), x_2(0), x_3(0)) = (1, 2, 3)$ .

Case 1.  $\alpha = 2$ . In this case, then  $h'(\cdot) \geq 1$ . By Theorem 1, consensus of (13) can be realized, and the decision value is  $\sum_{i=1}^3 \xi_i x_i(0) = 1/4 + 2/4 + 3/2 = 2.25$ , see Figure 1(a);

Case 2.  $\alpha = 0.5$ . In this case,  $h(\cdot)$  is not an increasing function, and consensus of (13) may not be realized, see Figure 1(b).

**Example 2:** In this simulation, we choose two protocols. One is the nonlinear protocol

$$h^*(x_i) = \begin{cases} x_i^2 & ; \text{ if } x_i > 1 \\ \sqrt{x_i} & ; \text{ if } 0 < x_i \leq 1 \\ -\sqrt{-x_i} & ; \text{ if } -1 < x_i \leq 0 \\ -x_i^2 & ; \text{ if } x_i \leq -1 \end{cases} \quad (14)$$

The other is the linear protocol

$$h(x_i) = x_i/2 \quad i = 1, 2, 3 \quad (15)$$

Simple calculations show that the derivative of  $h^*(\cdot)$  is no less than  $1/2$ . The consensus problem under the nonlinear protocol (14) can be realized faster than that under the linear one (15).



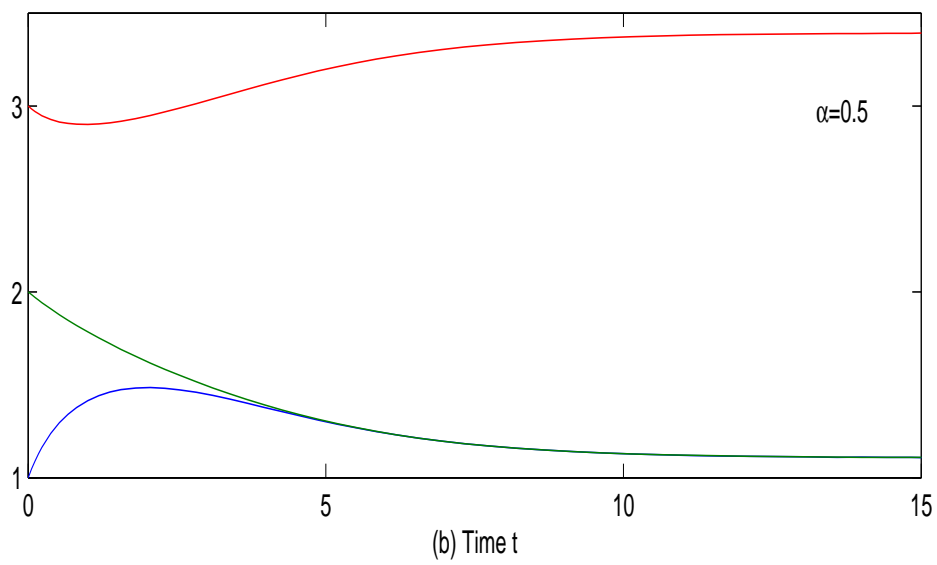
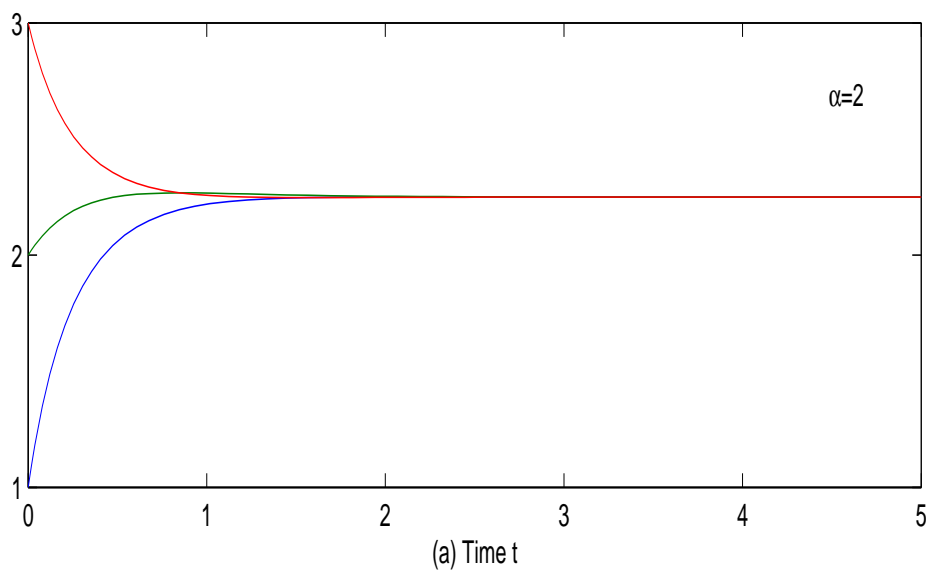


Figure 1: Consensus problem of (13) under different nonlinear functions

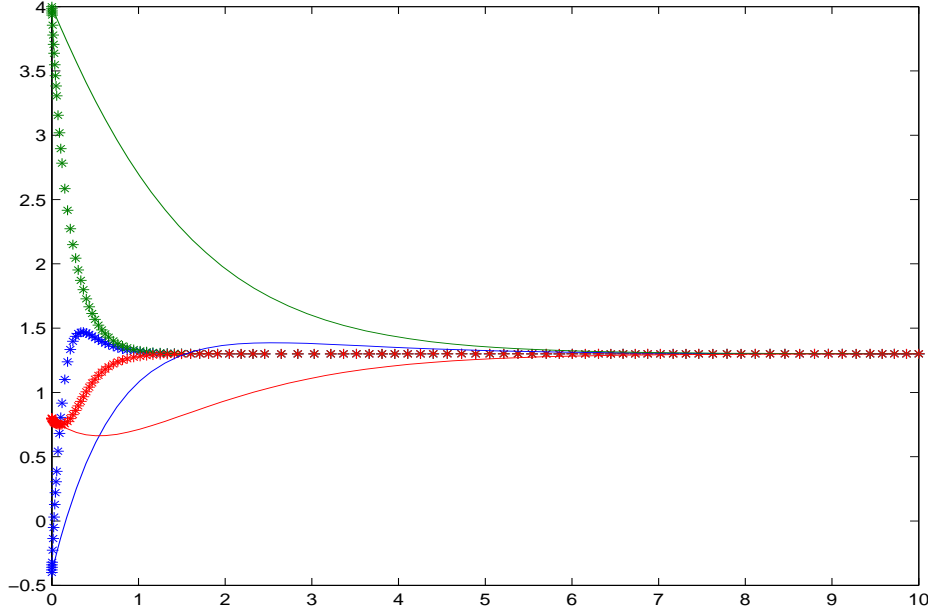


Figure 2: Consensus problem of (13) under nonlinear and linear protocols

In Figure 2, the dynamical behavior of the network (13) under the nonlinear protocol  $h^*(x)$  defined in (14) is displayed by line with star. Instead, for the linear protocol  $h(x)$  defined in (15), it is displayed by line without star. The initial value is chosen as:  $(x_1(0), x_2(0), x_3(0)) = (-0.4, 4, 0.8)$ . Simulations do show that consensus under the nonlinear protocol (14) is much faster than that under linear protocol (15).

## V Conclusions

In this paper, we investigate the consensus problem under nonlinear protocols. We generalize the results for undirected graphs to directed graphs. Moreover, our model can also be regarded as the generalization of consensus problem under linear protocols to nonlinear protocols. All the existing results with respect to consensus under linear protocols with directed/undirected graph and consensus under nonlinear protocols with undirected graph can be easily obtained by our approach. The convergence analysis is presented rigorously,

based on tools from algebraic graph theory, matrix theory, and control theory. Two simple examples are provided to show the effectiveness of the theoretical result.

## References

- [1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215-233, Jan. 2007.
- [2] R. Olfati-Saber, and R. M. Murray, "Consensus protocols for networks of dynamic agents," in *Proc. 2003 Am. Control Conf.*, 2003, pp. 951-956.
- [3] R. Olfati-Saber, and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520-1533, Sep. 2004.
- [4] W. Ren, and R. W. Beard, "Consensus seeking in multi-agent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655-661, May 2005.
- [5] V. Gazi, and K. M. Passino, "Stability analysis of swarms," *IEEE Trans. Autom. Control*, vol. 48, no. 4, pp. 692-697, Apr. 2003.
- [6] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: Algorithms and theory," *IEEE Trans. Autom. Control*, vol. 51, no. 3, pp. 401-420, Mar. 2006.
- [7] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*. Cambridge Univ. Press, 2001.
- [8] C. W. Wu, "Synchronization in networks of nonlinear dynamical systems coupled via a directed graph," *Nonlinearity*, vol. 18, pp. 1057-1064, Feb. 2005.
- [9] T. P. Chen, and Z. M. Zhu, "Exponential synchronization of nonlinear coupled dynamical networks," *Int. J. Bifur. Chaos*, vol. 17, no. 3, pp. 999-1005, 2007.

- [10] R. A. Horn, and C. R. Johnson, *Matrix Analysis*. Cambridge, U. K.: Cambridge Univ. Press, 1987.
- [11] R. K. Miller, and A. N. Michel, *Ordinary Differential Equations*. Academic Press, 1982.